Geometry of the three-qubit state, entanglement and division algebras

2003 J. Phys. A: Math. Gen. 36 8325

(http://iopscience.iop.org/0305-4470/36/30/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 8325–8339 PII: S0305-4470(03)59514-2

Geometry of the three-qubit state, entanglement and division algebras

Bogdan A Bernevig[1] **and Han-Dong Chen**[2]

[1] Department of Physics, Stanford University, Stanford, CA 94305, USA
[2] Department of Applied Physics, Stanford University, Stanford, CA 94305, USA

Received 11 February 2003, in final form 3 June 2003
Published 16 July 2003
Online at stacks.iop.org/JPhysA/36/8325

Abstract

We present a generalization to three qubits of the standard Bloch sphere representation for a single qubit and of the seven-dimensional sphere representation for two qubits presented in Mosseri *et al* (Mosseri R and Dandoloff R 2001 *J. Phys. A: Math. Gen.* **34** 10243). The Hilbert space of the three-qubit system is the 15-dimensional sphere S^{15}, which allows for a natural (last) Hopf fibration with S^8 as base and S^7 as fibre. A striking feature is, as in the case of one and two qubits, that the map is entanglement sensitive, and the two distinct ways of un-entangling three qubits are naturally related to the Hopf map. We define a quantity that measures the degree of entanglement of the three-qubit state. Conjectures on the possibility of generalizing the construction for higher qubit states are also discussed.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.67.Lx

1. Introduction

Quantum mechanics exhibits its difference from classical physical theories in many aspects. A quintessential property of quantum mechanics is quantum entanglement. Quantum entanglement rests at the centre of the applications such as quantum information and quantum computing. Maximally entangled Einstein–Podolsky–Rosen (EPR) pair [2] is an essential ingredient of teleportation [4], dense coding [3] and quantum key distribution [5, 6]. The maximally entangled three-qubit Greenberger–Horne–Zeilinger (GHZ) state [7] and the *m*-cat state are of cardinal importance to the applications such as cryptographic conferencing or superdense coding [8], quantum secret sharing or quantum information splitting [9]. Due to the entanglement of the Hilbert space states, it is a highly nontrivial problem to understand the properties of multi-qubit states. Recently, it has become clear [1] that the properties of the first two simplest qubit states, the single-qubit and the two-qubit states, are very deeply related to two very important mathematical objects, the first two Hopf fibrations $S^3 \xrightarrow{S^1} S^2$ and $S^7 \xrightarrow{S^3} S^4$. The global phase freedom of the single-qubit state and the entanglement which appears for the first time starting with the two-qubit case have been proved to be deeply related to the Hopf fibrations. For an entangled two-qubit state, performing a transformation on the first qubit space induces a transformation on the space of the second qubit space. This feature is naturally captured by the nontrivial second Hopf fibration. The Hopf fibration can determine if the two-qubit state is entangled or separable [1] and can also point to the degree of entanglement of a generic two-qubit pure state. Since obtaining a measure for the degree of entanglement is an essential issue of quantum computing, we believe it is extremely important if this method could be generalized to higher qubit states. Although attempts have been made towards describing the geometry of the three-qubit states [10, 11] (Mosseri *et al* [1] briefly mentioned the generalization of their construction to include the three-qubit state), to our knowledge, no complete description is available. In this paper, we generalize the discussion to the three-qubit state and the third Hopf fibration related to the last division algebra of the octonions. The entanglement is understood in a geometrical way and a quantitative measurement of entanglement is proposed. We describe the three-qubit Hilbert space as a nontrivial S^7 fibration over S^8. The entanglement quantity is proved to give the literature established values for the GHZ and Werner (W) states. The apparent failure of the algorithm for higher qubit states is also briefly discussed. We would like to stimulate discussion and progress on the proper n-qubit generalization as the rewards obtained from such a generalization could prove enormous, possibly leading to a full classification of entanglement. We want to mention that, as it stands, our discussion is applicable to pure states only.

The paper is organized as follows. In section 2 we briefly recall some well-known facts about the one-qubit state, the Bloch sphere representation and the close relation to the first Hopf fibration. In section 3 we present the recent results of Mosseri *et al* [1] which relate the two-qubit state to the second Hopf map ($S^7 \xrightarrow{S^3} S^4$). In section 4 we begin the treatment of the three-qubit state and convincingly prove that it is related to the third and last Hopf fibration thus clearly determining the geometry of the three-qubit state. We propose a quantity which can be used as a measure of the entanglement of the three-qubit state and comment on the prospective generalizations to higher qubit states. Although not strictly necessary, we use the language of the octonions, which nicely simplifies notation and points to very interesting and deep mathematical correspondences. In the appendix we give a brief introduction to the octonions and the three Hopf maps which we believe will be useful for a better understanding of the paper.

2. Single qubit, Bloch sphere and first Hopf fibration

The *pure* one-qubit state can be represented as a linear combination of up and down spins:

$$|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \qquad \alpha_0, \alpha_1 \in \mathbb{C} \qquad |\alpha_0|^2 + |\alpha_1|^2 = 1 \tag{1}$$

where we can parametrize

$$\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\exp\left(\mathrm{i}\frac{\phi}{2} + \mathrm{i}\frac{\chi}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)\exp\left(\mathrm{i}\frac{\phi}{2} - \mathrm{i}\frac{\chi}{2}\right)\end{pmatrix} \qquad \theta \in [0, \pi] \qquad \phi \in [0, 2\pi] \qquad \chi \in [0, 2\pi]. \tag{2}$$

The Hilbert space of a single qubit with fixed norm unity is the unit three-dimensional sphere S^3. But since quantum mechanics is $U(1)$ projective, the projective Hilbert space is defined up to a phase $\exp(\mathrm{i}\phi)$. Therefore the projective Hilbert space is $S^3/U(1) = S^3/S^1 = CP_1 = S^2$. This property points to a map between the full Hilbert space S^3 and the projective Hilbert space S^2, with the inverse map (fibre) being an S^1. This map is the well-known first Hopf map, $S^3 \xrightarrow{S^1} S^2$ which gives S^3 as an S^1 fibration over a base space S^2, the first in a series of maps that are deeply related to the structure of consistently defined number structures (division algebras, see the appendix). The map has the explicit form

$$h_1 : \begin{array}{ccc} \mathbb{C} \otimes \mathbb{C} & \longrightarrow & \mathbb{C} \cup \{\infty\} \approx S^2 \\ (\alpha_0, \alpha_1) & \longrightarrow & h_1 = \alpha_0 \alpha_1^{-1} \end{array} \qquad |\alpha_0|^2 + |\alpha_1|^2 = 1 \tag{3a}$$

$$h_2 : \begin{array}{ccc} \mathbb{C} \cup \{\infty\} & \longrightarrow & S^2 \\ h_1 & \longrightarrow & X_i \ (i = 1, 2, 3) \end{array} \qquad \sum_{i=1}^{3} X_i^2 = 1 \tag{3b}$$

$$h_2 \circ h_1(\alpha_0, \alpha_1) = X_i = \langle \sigma_i \rangle_\Psi = (\alpha_0^\star, \alpha_1^\star) \sigma_i \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{3c}$$

where σ_i $(i = 1, 2, 3)$ are the three Pauli matrices. We can clearly see that the X_i are defined up to a $U(1)$ ambiguity in α_0, α_1. This map is very useful in describing the density matrix for one qubit. The most general form of this matrix is

$$\rho = \frac{1}{2}(I + X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3) = \frac{1}{2}\begin{pmatrix} 1 + X_3 & X_1 - \mathrm{i}X_2 \\ X_1 + \mathrm{i}X_2 & 1 - X_3 \end{pmatrix} \tag{4}$$

with the constraint $\det \rho = 1 - X_1^2 - X_2^2 - X_3^2 \geqslant 0$. For *pure* qubit states, $\det \rho = 0$. The complete description of the single-qubit Hilbert space and its essential phase freedom can therefore be understood through the first Hopf fibration. This fibration is nontrivial since $S^3 \neq S^1 \otimes S^2$. Physically, this means that it is impossible to consistently ascribe a definite phase to each point on the Bloch sphere.

3. Two qubits, entanglement and the second Hopf fibration

This section summarizes the results of Mosseri *et al* [1]. A *pure* two-qubit state reads

$$|\Psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \beta_0|10\rangle + \beta_1|11\rangle \tag{5a}$$

$$\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{C} \qquad |\alpha_0|^2 + |\alpha_1|^2 + |\beta_0|^2 + |\beta_1|^2 = 1. \tag{5b}$$

The normalization condition means the Hilbert space of the two-qubit state with fixed norm unity is a seven-dimensional sphere S^7 and the projective Hilbert space is $S^7/U(1) = CP_3$. The Hilbert space ε is the tensorial product of single-qubit Hilbert spaces $\varepsilon_1 \otimes \varepsilon_2$. In general, performing a transformation on the first qubit space induces a transformation on the space of the second qubit space. However, for the case in which $\alpha_0\beta_1 = \alpha_1\beta_0$ one can independently transform the spaces of the two single qubits. We then call the state non-entangled or separable. To gain insight into the geometry and structure of the two-qubit we need to analyse the S^7 manifold of the Hilbert space. S^7 can be parametrized in many different ways as a product of manifolds, but the most interesting parametrization [1] is as an S^3 fibre over an S^4. Notation can be greatly simplified by introducing a pair of quaternions:

$$q_1 = \alpha_0 + \alpha_1 \mathrm{i}_2 = \mathrm{Re}\,\alpha_0 + \mathrm{i}_1\,\mathrm{Im}\,\alpha_0 + \mathrm{i}_2\,\mathrm{Re}\,\alpha_1 + \mathrm{i}_3\,\mathrm{Im}\,\alpha_1 \tag{6a}$$

$$q_2 = \beta_0 + \beta_1 \mathrm{i}_2 = \mathrm{Re}\,\beta_0 + \mathrm{i}_1\,\mathrm{Im}\,\beta_0 + \mathrm{i}_2\,\mathrm{Re}\,\beta_1 + \mathrm{i}_3\,\mathrm{Im}\,\beta_1. \tag{6b}$$

$\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3$ are square roots of -1 and form a basis for the imaginary part of the quaternionic space ($\mathbb{Q} \sim \mathbb{R}^4$, $\mathrm{Im}\,\mathbb{Q} \sim R^3$, see the appendix). Similar to the single-qubit case, we can now define the following map:

$$h_1 : \begin{array}{ccc} \mathbb{Q} \otimes \mathbb{Q} & \longrightarrow & \mathbb{Q} \cup \{\infty\} \approx S^4 \\ (q_1, q_2) & \longrightarrow & h_1 = q_1 q_2^{-1} \end{array} \qquad |q_1|^2 + |q_2|^2 = 1 \tag{7a}$$

$$h_2 : \begin{array}{ccc} \mathbb{Q} \cup \{\infty\} & \longrightarrow & S^4 \\ h_1 & \longrightarrow & X_i \; (i = 1, \ldots, 5) \end{array} \qquad \sum_{i=1}^{5} X_i^2 = 1 \tag{7b}$$

$$h_2 \circ h_1(q_1, q_2) = X_i = \langle \sigma_i \rangle_\Psi = (q_1^\star, q_2^\star)\sigma_i \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{7c}$$

where

$$\begin{array}{lll} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_2 = \begin{pmatrix} 0 & \mathrm{i}_1 \\ -\mathrm{i}_1 & 0 \end{pmatrix} & \sigma_3 = \begin{pmatrix} 0 & \mathrm{i}_2 \\ -\mathrm{i}_2 & 0 \end{pmatrix} \\ \sigma_4 = \begin{pmatrix} 0 & \mathrm{i}_3 \\ -\mathrm{i}_3 & 0 \end{pmatrix} & \sigma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \end{array} \tag{8}$$

are a generalization of the Pauli matrices to quaternionic space.

The points (q_1, q_2), and $(q_1 q, q_2 q)$, where q is a unit quaternion (S^3), are mapped onto the same point of the base space S^4 and therefore the map is a nontrivial fibration $S^7 \stackrel{S^3}{\longrightarrow} S^4$. This fibration is entanglement sensitive [1] in the sense that the separable states defined by $\alpha_0\beta_1 = \alpha_1\beta_0$ will be mapped onto the subset of pure complex numbers in the quaternion field, i.e.,

$$X_3\big|_{\alpha_0\beta_1=\alpha_1\beta_0} = X_4\big|_{\alpha_0\beta_1=\alpha_1\beta_0} = 0 \qquad \text{or} \qquad h_1(q_1, q_2)\big|_{\alpha_0\beta_1=\alpha_1\beta_0} \in \mathbb{C} \subset \mathbb{Q}. \tag{9}$$

It follows that the base space simplifies to an S^2 for non-entangled (separable) qubits. The partially traced density matrix ρ_1 can be written as a functional of the variables in the base space [1]:

$$\rho_1 = \mathrm{Tr}_2\rho = [I_1 \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)_2]|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 + X_5 & x_1 - \mathrm{i}X_2 \\ X_1 + \mathrm{i}X_2 & 1 - X_5 \end{pmatrix}. \tag{10}$$

This is the most general density matrix for a one-qubit system. The Bloch ball for one qubit is then recovered from the two-qubit system by the partial trace. The determinant of ρ_1 is

$$\det \rho_1 = 1 - X_1^2 - X_2^2 - X_5^2 = X_3^2 + X_4^2. \tag{11}$$

$\det \rho_1 = 0$ for non-entangled qubits. Therefore, the density matrix ρ_1 represents a pure state if $|\Psi\rangle$ is non-entangled. Otherwise, ρ_1 represents a mixed state. Mathematically, losing the information of the second qubit means integrating out or partial tracing the degree of freedom of the second qubit. Then the resulting density matrix is only related to the base space. It then follows naturally that the information of the second qubit is stored in the fibre space while the information of the first qubit and the correlation between these two qubits is stored in the base space. In the non-entangled case, the first Hopf map can be applied to the fibre S^3 (Hilbert space of the second qubit) as described in the previous section. This would mod out the phase degree of the freedom. Finally, the S^7 fibration simplifies to $S^2 \otimes S^2$ for non-entangled qubits, with one S^2 from the base and the other one from the fibre. In addition, the quantity $X_3^2 + X_4^2$ might be useful to quantitatively measure the entanglement [1].

4. Three qubits, entanglement and the third Hopf fibration

It is interesting to see that the one-qubit and two-qubit systems are closely related to the first two Hopf fibrations and the division algebras of the complex numbers and the quaternions. This relation points to both insightful comments on the geometry of the Hilbert space, and quantities which might describe entanglement. The two-qubit system is the only system for which entanglement problem has so far been solved [12] Multiple complications arise for higher qubit problems [13, 14]. In this section we go one step further to the first complicated qubit state, the three qubits. We show that its Hilbert space geometry can be closely related to the geometry of the third and last Hopf fibration and prove several insightful relations on the entanglement of such state.

4.1. The three-qubit Hilbert space. two-qubit ⊗ one-qubit entanglement

The Hilbert space for the three-qubit is the tensor product of the one-qubit Hilbert spaces $\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3$ with a direct product basis: $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$. A pure three-qubit state reads

$$|\Psi\rangle = \alpha_0|000\rangle + \alpha_1|001\rangle + \beta_0|010\rangle + \beta_1|011\rangle + \delta_0|100\rangle + \delta_1|101\rangle + \gamma_0|110\rangle + \gamma_1|111\rangle \quad (12a)$$

$$\begin{aligned} &\alpha_0, \alpha_1, \beta_0, \beta_1, \delta_0, \delta_1, \gamma_0, \gamma_1 \in \mathbb{C} \\ &|\alpha_0|^2 + |\alpha_1|^2 + |\beta_0|^2 + |\beta_1|^2 + |\delta_0|^2 + |\delta_1|^2 + |\gamma_0|^2 + |\gamma_1|^2 = 1. \end{aligned} \quad (12b)$$

Differently from the case of two qubits, there are now two ways in which the three-qubit state can be separated. In the first case, the three-qubit case can be separated in the subspace of a single qubit with basis $\{|0\rangle, |1\rangle\}$ and the subspace of two-qubit $\{|00\rangle, |01\rangle, |10\rangle|11\rangle\}$:

$$|\Psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|00\rangle + d|01\rangle + e|10\rangle + f|11\rangle) \quad (13a)$$

$$a, b, c, d, e, f \in \mathbb{C} \qquad (|a|^2 + |b|^2)(|c|^2 + |d|^2 + |e|^2 + |f|^2) = 1. \quad (13b)$$

In this scenario, we get the following relations:

$$\begin{aligned} &\alpha_0\gamma_1 = \delta_0\beta_1 \qquad \alpha_0\gamma_0 = \delta_0\beta_0 \qquad \alpha_0\delta_1 = \delta_0\alpha_1 \\ &\alpha_1\gamma_1 = \delta_1\beta_1 \qquad \alpha_1\gamma_0 = \delta_1\beta_0 \qquad \beta_0\gamma_1 = \gamma_0\beta_1. \end{aligned} \quad (14)$$

Among these six conditions, only four are fundamental, from which the other two can be obtained.

We can also go one step further and separate the two-qubit subspace. In this case, the three-qubit state becomes fully separated in the three one-qubit subspaces.

$$|\Psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \otimes (e|0\rangle + f|1\rangle) \quad (15a)$$

$$a, b, c, d, e, f \in \mathbb{C} \qquad (|a|^2 + |b|^2)(|c|^2 + |d|^2)(|e|^2 + |f|^2) = 1. \quad (15b)$$

The first step towards separating the three-qubit space is the partial one-qubit ⊗ two-qubit separation.

The normalization condition (12*b*) for the general three-qubit state identifies its Hilbert space with the 15-dimensional sphere S^{15}. This manifold can be parametrized in many ways, but considering the experience of the two previous sections and reminding ourselves of the existence of a third and last Hopf fibration $S^{15} \xrightarrow{S^7} S^8$, it is tempting to see whether it plays a role in the Hilbert space description.

4.2. Octonionic representation of three-qubit state and the third Hopf fibration

The most aesthetic way to introduce this fibration is with the use of octonions instead of quaternions or complex numbers. Using octonions introduces complications since they are not only non-commutative (like the quaternions) but also non-associative (see the appendix). However, we feel that this discomfort is compensated by the fact that the mathematics becomes very compact and the connection with division algebra and the Cayley–Dickson construction (see the appendix) becomes much clearer.

The construction of the two octonions from the complex coefficients of the three-qubit state in equation (12*a*) proceeds as follows: we first define four quaternions:

$$\begin{aligned} q_1 &= (\alpha_0, \alpha_1) = \alpha_0 + \alpha_1 \mathrm{i}_2 \qquad & q_2 &= (\beta_0, \beta_1) = \beta_0 + \beta_1^\star \mathrm{i}_2 \\ q_3 &= (\delta_0, \delta_1) = \delta_0 + \delta_1 \mathrm{i}_2 \qquad & q_4 &= (\gamma_0, \gamma_1) = \gamma_0 + \gamma_1^\star \mathrm{i}_2. \end{aligned} \tag{16}$$

They satisfy the normalization $|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2 = 1$. Out of these four quaternions, by the Cayley–Dickson construction we can create two octonions belonging to the eight-dimensional octonionic space $\mathbb{O} \sim R^8$:

$$o_1 = (q_1, q_2) = q_1 + q_2\, \mathrm{i}_4 \qquad o_2 = (q_3, q_4) = q_3 + q_4\, \mathrm{i}_4. \tag{17}$$

The normalization condition now translates into $|o_1|^2 + |o_2|^2 = 1$, parametrizing an S^{15}. $\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3, \mathrm{i}_4$ generate through multiplications $\mathrm{i}_5, \mathrm{i}_6, \mathrm{i}_7$. These seven imaginary square roots of -1, along with the unity, close the octonionic multiplication table (see the appendix). The choice in the definition of the four quaternions is specifically related to the tensor-product nature of the three-qubit Hilbert space. Had we made a different choice for the four quaternions (two octonions), we would have induced an anisotropy on S^{15}, much in the same case as in Mosseri *et al* [1]. The Hopf map from S^{15} to S^8 can again be described as a map h_1 from $\mathbb{O} \otimes \mathbb{O}$ to $\mathbb{O} \cup \infty$ composed with an inverse stereographic map h_2 from $\mathbb{O} \cup \infty$ to S^8:

$$h_1: \begin{matrix} \mathbb{O} \otimes \mathbb{O} & \longrightarrow & \mathbb{O} \cup \{\infty\} \approx S^8 \\ (q_1, q_2) & \longrightarrow & h_1 = o_1 o_2^{-1} \end{matrix} \qquad |o_1|^2 + |o_2|^2 = 1 \tag{18a}$$

$$h_2: \begin{matrix} \mathbb{O} \cup \{\infty\} & \longrightarrow & S^8 \\ h_1 & \longrightarrow & X_i\ (i = 1, \ldots, 9) \end{matrix} \qquad \sum_{i=1}^{9} X_i^2 = 1 \tag{18b}$$

$$h_2 \circ h_1(o_1, o_2) = X_i = \langle \sigma_i \rangle_\Psi = (o_1^\star, o_2^\star)\sigma_i \begin{pmatrix} o_1 \\ o_2 \end{pmatrix} \tag{18c}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_{2,3,4,5,6,7,8} = \begin{pmatrix} 0 & \mathrm{i}_{1,2,3,4,5,6,7} \\ -\mathrm{i}_{1,2,3,4,5,6,7} & 0 \end{pmatrix} \qquad \sigma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

are a generalization of the Pauli matrices to octonionic space. As in the case of previous Hopf maps, the fibration is not trivial, as the space S^8 is not embedded in S^{15}. The fibre is a seven-dimensional sphere S^7, as can be seen by taking the inverse map:

$$h_1^{-1}(y) = \begin{pmatrix} \{(yd, d) \mid d \in \mathbb{O}, |yd, d| = 1\}, x \neq \infty \\ \{(c, 0) \mid c \in \mathbb{O}, |c| = 1\}, x = \infty \end{pmatrix}. \tag{20}$$

We need to pause for a second and address an important comment. Although in the case of quaternions which are only non-commutative, it was clear that the map would have a unit quaternion (S^3) as fibre, in the case of octonions, because of their non-associativity, this is not automatically transparent. However, the fact that the algebra is still alternative (see the appendix) (no other higher dimensional alternative algebra is known) comes to our rescue and renders the fibre of the map be a unit octonion S^7.

The first interesting feature of the fibration is revealed upon explicit computation.

$$h_1(o_1, o_2) = o_1 o_2^{-1} = \frac{C_1 + C_2\, \mathrm{i}_2 + C_3\, \mathrm{i}_4 + C_4^\star\, \mathrm{i}_6}{|\delta_0|^2 + |\delta_1|^2 + |\gamma_0|^2 + |\gamma_1|^2} \tag{21}$$

with

$$C_1 = \alpha_0\delta_0^\star + \delta_1^\star\alpha_1 + \gamma_0^\star\beta_0 + \beta_1\gamma_1^\star \tag{22a}$$

$$C_2 = \alpha_1\delta_0 - \delta_1\alpha_0 + (\beta_1\gamma_0 - \gamma_1\beta_0)^\star \tag{22b}$$

$$C_3 = \beta_0\delta_0 - \gamma_0\alpha_0 + (\alpha_1\gamma_1 - \delta_1\beta_1)^\star \tag{22c}$$

$$C_4 = \delta_1\beta_0 - \alpha_1\gamma_0 + (\beta_1\delta_0 - \gamma_1\alpha_0)^\star. \tag{22d}$$

For the generic three-qubit state, the h_1 map is octonionic in nature, as we see above. However, for the case in which the three-qubit state is separable as a one-qubit $\otimes$ two-qubit, the h_1 maps into the subspace of pure complex numbers $\mathbb{C} \cup \infty$ in the octonionic field $\mathbb{O} \cup \infty$:

$$h(o_1, o_2)\bigg|_{3=1\otimes 2} = \frac{C_1}{|\delta_0|^2 + |\delta_1|^2 + |\gamma_0|^2 + |\gamma_1|^2} \in \mathbb{C} \cup \infty. \tag{23}$$

We have just proved that the last Hopf map is entanglement sensitive. In other words, by computing the value of the map one can establish whether the three-qubit state is entangled or is separable as a one-qubit $\otimes$ two-qubit state. We will come back to this later on as we define a quantity that characterizes the degree of entanglement and we will see that the separated two-qubit state lives on the fibre of the map while the one-qubit state lives on the base space of the map. The next step in analysing the geometry of the Hilbert space consists of an analysis of the base space. For future reference, we give here the expressions of the coordinates on the base space S^8:

$$X_1 = o_1 o_2^\star + o_2 o_1^\star \tag{24a}$$

$$X_2 = \mathrm{Re}[\mathrm{i}_1(o_1 o_2^\star - o_2 o_1^\star)] \tag{24b}$$

$$X_3 = \mathrm{Re}[\mathrm{i}_2(o_1 o_2^\star - o_2 o_1^\star)] \tag{24c}$$

$$X_4 = \mathrm{Re}[\mathrm{i}_3(o_1 o_2^\star - o_2 o_1^\star)] \tag{24d}$$

$$X_5 = \mathrm{Re}[\mathrm{i}_4(o_1 o_2^\star - o_2 o_1^\star)] \tag{24e}$$

$$X_6 = \mathrm{Re}[\mathrm{i}_5(o_1 o_2^\star - o_2 o_1^\star)] \tag{24f}$$

$$X_7 = \mathrm{Re}[\mathrm{i}_6(o_1 o_2^\star - o_2 o_1^\star)] \tag{24g}$$

$$X_8 = \mathrm{Re}[\mathrm{i}_7(o_1 o_2^\star - o_2 o_1^\star)] \tag{24h}$$

$$X_9 = o_1 o_1^\star - o_2 o_2^\star \tag{24i}$$

where $o_1 o_2^\star - o_2 o_1^\star$ is purely imaginary and $o_1 o_2^\star + o_2 o_1^\star$ and $o_1 o_1^\star - o_2 o_2^\star$ are purely real. Their values are

$$o_1 o_1^\star - o_2 o_2^\star = \alpha_0\alpha_0^\star + \alpha_1\alpha_1^\star + \beta_0\beta_0^\star + \beta_1^\star\beta_1 - \delta_0\delta_0^\star - \delta_1\delta_1^\star - \gamma_0\gamma_0^\star - \gamma_1^\star\gamma_1 \tag{25a}$$

$$o_1 o_2^\star + o_2 o_1^\star = \delta_0^\star\alpha_0 + \delta_1^\star\alpha_1 + \gamma_0^\star\beta_0 + \beta_1\gamma_1^\star + \delta_0\alpha_0^\star + \delta_1\alpha_1^\star + \gamma_0\beta_0^\star + \beta_1^\star\gamma_1 \tag{25b}$$

$$o_1 o_2^\star - o_2 o_1^\star = ((a_0, a_1), (b_0, b_1)) \qquad a_0, a_1, b_0, b_1 \in \mathbb{C} \tag{25c}$$

with

$$a_0 = \delta_0^\star\alpha_0 + \delta_1^\star\alpha_1 + \gamma_0^\star\beta_0 + \beta_1\gamma_1^\star - \delta_0\alpha_0^\star - \delta_1\alpha_1^\star - \gamma_0\beta_0^\star - \beta_1^\star\gamma_1 \tag{26a}$$

$$a_1 = 2\alpha_1\delta_0 - 2\delta_1\alpha_0 + 2\beta_1^\star\gamma_0^\star - 2\gamma_1^\star\beta_0^\star \tag{26b}$$

$$b_0 = 2\beta_0\delta_0 - 2\gamma_0\alpha_0 + 2\alpha_1^\star\gamma_1^\star - 2\delta_1^\star\beta_1^\star \tag{26c}$$

$$b_1 = 2\delta_1\beta_0 - 2\alpha_1\gamma_0 + 2\beta_1^\star\delta_0^\star - 2\gamma_1^\star\alpha_0^\star. \tag{26d}$$

The nine coordinates (subject to one constraint) of the S^8 represent the generalization of the Bloch sphere representation. For the case when the three-qubit state is separable as a one-qubit ⊗ two-qubit state, the map becomes purely complex, as we have shown. In this case,

$$o_1 o_2^\star - o_2 o_1^\star = \delta_0^\star \alpha_0 + \delta_1^\star \alpha_1 + \gamma_0^\star \beta_0 + \beta_1 \gamma_1^\star - \delta_0 \alpha_0^\star - \delta_1 \alpha_1^\star - \gamma_0 \beta_0^\star - \beta_1^\star \gamma_1 \tag{27}$$

which means $X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$. This implies that only an $S^2\left(X_1^2 + X_2^2 + X_9^2\right)$ in the base space S^8 is used in the separable case. Therefore now things become clear: for a generic three-qubit state, the Hilbert space is a 15-dimensional sphere S^{15}. This sphere admits many parametrizations, the most famous of which is the third and last Hopf map expressible as an S^7 fibration over S^8. As we have shown, this fibration is entanglement sensitive, in the sense that it can detect whether the three-qubit state is separable as a product of a one-qubit state and a two-qubit state. Moreover, an analysis of where the states are located points out that the two-qubit state occupies the fibre S^7 of the map while the single-qubit state occupies three (X_1, X_2, X_9) of the nine coordinates on the base space S^8. The rest of the coordinates somehow characterize the degree of the entanglement between these two states, such that they are zero—as shown—in the case when the three-qubit states are totally separable as a one-qubit ⊗ two-qubit state. Quantifying the degree of the entanglement will be our next priority. Since we have now established where the two-qubit and the single-qubit states live, we now have a very similar picture to that developed by Mosseri *et al* [1]. To obtain the fully separable three-qubit state into three one-qubit states, we first separate it into a one-qubit ⊗ two-qubit state $S^2 \otimes S^7$. We then focus on the fibre of the map, and use the second Hopf fibration to separate it into an $S^2 \otimes S^3$ as shown in the previous sections. We can then mod out the phase degree of freedom by again particularizing to the fibre of the second Hopf fibration and using the first Hopf fibration to mod out an $S^1 = U(1)$.

4.3. Discussion

Let us now obtain the general expression for a state Ψ_O ($\in S^{15}$) which is sent to O by the map h_1. The inverse of the third Hopf map gives

$$\Psi_O = (\cos\Omega \exp(-\Theta \mathbf{T}/2) o, \sin\Omega \exp(\Theta \mathbf{T}/2) o) \tag{28}$$

where $\cos\Theta = S(O')$, $\sin\Omega = X_1/S(O')$, o is a unit octonion which spans the S^7 fibre and $\mathbf{T}$ is a unit pure imaginary octonion

$$\mathbf{T} = \frac{1}{\sin\Theta}\left(\sum_{m=1}^{7} \mathbf{V}_m(O') i_m\right). \tag{29}$$

Here $S(O') = (O' + (O')^*)/2$ and $\mathbf{V}(O') = (O' - (O')^*)/2$ are the scalar and vectorial parts of $O' \equiv O/|O|$.

4.3.1. Separable states.

If the first qubit can be separated from the other two, O is a complex number. Consequently, the state Ψ_Q becomes

$$\Psi_O = (\cos\Omega \exp(-\Theta \mathrm{i}/2) o, \sin\Omega \exp(\Theta \mathrm{i}/2) o). \tag{30}$$

The S^8 base space reduces to S^2 sphere since $X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$. This S^2 sphere is exactly the Bloch sphere of the first qubit.

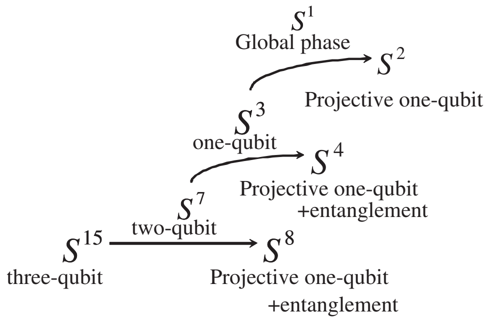

Figure 1. The iteration of Hopf fibration: three-qubit $\longrightarrow$ one-qubit $\otimes$ two-qubit $\longrightarrow$ one-qubit $\otimes$ one-qubit $\otimes$ one-qubit.

For the second and third qubits described by $o \in S^7$, we can define the coordinate system on fibre as

$$|00\rangle_O = (\cos\Omega \exp(-\mathrm{i}\Theta/2)|0\rangle_1 + \sin\Omega \exp(\mathrm{i}\Theta/2)|1\rangle_1) \otimes (|0\rangle_2 \otimes |0\rangle_3) \tag{31a}$$

$$|01\rangle_O = (\cos\Omega \exp(-\mathrm{i}\Theta/2)|0\rangle_1 + \sin\Omega \exp(\mathrm{i}\Theta/2)|1\rangle_1) \otimes (|0\rangle_2 \otimes |1\rangle_3) \tag{31b}$$

$$|10\rangle_O = (\cos\Omega \exp(-\mathrm{i}\Theta/2)|0\rangle_1 + \sin\Omega \exp(\mathrm{i}\Theta/2)|1\rangle_1) \otimes (|1\rangle_2 \otimes |0\rangle_3) \tag{31c}$$

$$|11\rangle_O = (\cos\Omega \exp(-\mathrm{i}\Theta/2)|0\rangle_1 + \sin\Omega \exp(\mathrm{i}\Theta/2)|1\rangle_1) \otimes (|1\rangle_2 \otimes |1\rangle_3). \tag{31d}$$

A generic state Ψ_O in the S^7 fibre can be decomposed as

$$|\Psi_O\rangle = A_0|00\rangle_O + A_1|01\rangle_O + B_0|10\rangle_O + B_1|11\rangle_O \tag{32}$$

with $A_0, A_1, B_0, B_1 \in \mathbb{C}$ and $|A_0|^2 + |A_1|^2 + |B_0|^2 + |B_1|^2 = 1$. It is straightforward to see that the three-qubit system reduces to one-qubit $\otimes$ two-qubit. Now, we can fibrate the S^7 fibre space using the second Hopf map for this four-level two-qubit system. If this two-qubit is separable, the S^7 fibre space itself reduces to $S^2 \otimes S^3$ with S^3 living on the fibre. Then we can again fibrate the S^3 to mod out the global phase. Consequently, if it is fully separable, the three-qubit reduces to $S^2 \otimes S^2 \otimes S^2$ with the first, second and third qubits living in the base space of the S^{15} fibration, the base space of the S^7 fibration of the fibre and the fibre of S^7 fibration of the fibre, respectively. Figure 1 sketches the iteration of the three Hopf fibrations.

4.3.2. Entangled states. Now, let us turn to the maximally entangled states (MES). They corresponding to the vector $C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_3 + C_4\,\mathrm{i}_6$ have maximal norm. For a MES, $|\Psi_O\rangle$ reads

$$|\Psi_O\rangle = \frac{1}{\sqrt{2}}\left(\exp\left(-\pi\frac{C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_4 + C_4^{\star}\,\mathrm{i}_6}{2}\right)o, \exp\left(\pi\frac{C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_4 + C_4^{\star}\,\mathrm{i}_6}{2}\right)o\right). \tag{33}$$

The MES expands a five-dimensional sphere S^5. For $C_4 = \pm\frac{1}{2}$ and $o = (1 \pm \mathrm{i}_6)/\sqrt{2}$, the standard GHZ state is obtained from equation (33).

For any ΩMES, $|\Psi_O\rangle$ can be written as

$$|\Psi_O\rangle = \left(\cos\Omega\exp\left(-\frac{\pi}{4}\frac{C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_4 + C_4^{\star}\,\mathrm{i}_6}{|C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_3 + C_4^{\star}\,\mathrm{i}_6|}\right)o, \sin\Omega\exp\left(\frac{\pi}{4}\frac{C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_4 + C_4^{\star}\,\mathrm{i}_6}{|C_2\,\mathrm{i}_2 + C_3\,\mathrm{i}_3 + C_4^{\star}\,\mathrm{i}_6|}\right)o\right). \tag{34}$$

From equation (41), one sees the fact that the base space contains the information of the first qubit and the information of the correlation between it and the other two qubits while the

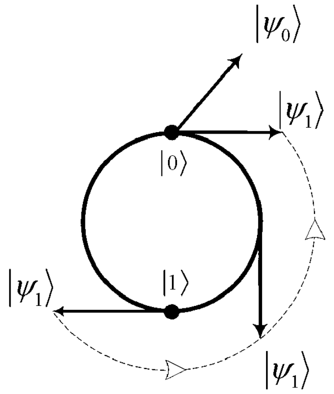

Figure 2. The graphical representation of the decomposition given by equation (36). After parallel transporting the vector $|\Psi_1\rangle$ from the south pole to the north pole, we can define an angle to quantify the difference between $|\Psi_0\rangle$ and $|\Psi_1\rangle$. This angle could be used to quantify the entanglement of the state $|\Psi\rangle$.

fibre only contains the information of the second and third qubits only. We can utilize this observation to generalize the Bloch sphere representation. The Hopf map clearly suggests splitting the representation into a product of base and fibre subspaces. For the base space S^8, we propose to only keep three coordinates:

$$(X_1, X_2, X_9) = (\langle\sigma_x \otimes I_{\text{two-qubit}}\rangle, \langle\sigma_y \otimes I_{\text{two-qubit}}\rangle, \langle\sigma_z \otimes I_{\text{two-qubit}}\rangle). \tag{35}$$

All states are then mapped onto a ball B^3 of radius 1 described by $0 \leqslant X_1^2 + X_2^2 + X_9^2 \leqslant 1$. The set of separable states is mapped onto the S^2 boundary as discussed previously. The centre of the ball corresponds to MES. The concentric spherical shells correspond to the set of states with the same entanglement as defined in equation (41).

4.3.3. Angle description of entanglement. For a generic three-qubit state given by equation (12*a*), we can decompose it as

$$|\Psi\rangle = |0\rangle|\Psi_0\rangle + |1\rangle|\Psi_1\rangle \tag{36}$$

with

$$|\Psi_0\rangle = A_0|00\rangle + A_1|01\rangle + B_0|10\rangle + B_1|11\rangle \tag{37a}$$

$$|\Psi_1\rangle = A'_0|00\rangle + A'_1|01\rangle + B'_0|10\rangle + B'_1|11\rangle. \tag{37b}$$

Geometrically, we can imagine that $|\Psi_0\rangle$ ($|\Psi_1\rangle$) lives on the north (south) pole of the one-qubit Bloch sphere as sketched in figure 2. After parallel transporting the vector $|\Psi_1\rangle$ from the south pole to the north pole, we can define an angle to quantify the difference between $|\Psi_0\rangle$ and $|\Psi_1\rangle$. If these two vectors are pointing in the same direction, i.e.,

$$|\Psi_0\rangle = C|\Psi_1\rangle \qquad (C \in \mathbb{C}) \tag{38}$$

the first qubit can be separated from the other two and the three-qubit state $|\Psi\rangle$ reduces to a one-qubit $\otimes$ two-qubit state. We then can iterate this decomposition for the two-qubit state $|\Psi_0\rangle$.

The condition (38) leads to the same conditions as given in equation (14). A natural definition of the entanglement is then given by

$$E = A \sum_{b,c,b'c'} \left[|t_{0bc}t_{1b'c'} - t_{0b'c'}t_{1bc}|^2 + |t_{b0c}t_{b'1c'} - t_{b'0c'}t_{b1c}|^2 + |t_{b0c}t_{b'1c'} - t_{b'0c'}t_{b1c}|^2\right] \tag{39}$$

where A is a proper normalization factor and $t_{abc} \equiv \langle abc|\Psi\rangle$. The generalization of this definition is straightforward. This definition is exactly the same as that discussed by Meyer *et al* [15].

4.3.4. Quantifying entanglement. Based on our above discussion, we are now in a position to propose a quantity that quantifies the degree of entanglement of a three-qubit state. As our discussion so far suggests, we need to probe for the entanglement of three qubits in a one-qubit ⊗ two-qubit state. (Subsequently, we can particularize to the fibre of the third Hopf map and classify the degree of entanglement of the two-qubit state.) We therefore partially trace two qubits to obtain the partially traced matrix ρ_1:

$$\rho_1 = \frac{1}{2}\begin{pmatrix} 1 + X_9 & X_1 - \mathrm{i}X_2 \\ X_1 + \mathrm{i}X_2 & 1 - X_9 \end{pmatrix}. \tag{40}$$

Usually, for generic three-qubit states, $\det\rho_1 > 0$. However, in the case of two-qubit ⊗ one-qubit entanglement, the determinant of the matrix vanishes, and therefore $X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0$. It seems obvious then that we could use the quantity

$$E = X_3^2 + X_4^2 + X_5^2 + X_6^2 + X_7^2 + X_8^2 = 1 - X_1^2 - X_2^2 - X_9^2 \tag{41}$$

to quantify entanglement. Small values of E means high degree of separability of one-qubit and two-qubit Hilbert spaces in the three-qubit state and vice versa. Note that this quantity E only measures the entanglement between the first qubit and the two-qubit system of second and third qubits. Similarly, we can construct the second or third qubit into the base space to get two different constructions of the Hopf fibration. A more reasonable definition of the measurement of the entanglement will be the average of the quantity given in equation (41) over all possible constructions.

Let us now test these assumptions on two well-known states, GHZ and W states of the three-qubit problem. The generalized GHZ states read

$$|\mathrm{GHZ}\rangle|_{\text{generalized}} = \alpha_0|000\rangle + \gamma_1|111\rangle \quad\Longrightarrow\quad \begin{aligned} X_5 &= 2\,\mathrm{Re}\,\gamma_1\,\mathrm{Im}\,\alpha_0 + 2\,\mathrm{Im}\,\gamma_1\,\mathrm{Re}\,\alpha_0 \\ X_6 &= 2\,\mathrm{Re}\,\gamma_1\,\mathrm{Re}\,\alpha_0 - 2\,\mathrm{Im}\,\gamma_1\,\mathrm{Im}\,\alpha_0 \\ X_9 &= |\alpha_0|^2 - |\gamma_1|^2 \end{aligned} \tag{42}$$

the other X being zero, and with a degree of entanglement $E = 1 - |\alpha_0|^2 + |\gamma_1|^2$. For the *pure* GHZ state $\alpha = \gamma_1 = 1/\sqrt{2}$ and therefore $E = 1$, meaning that the GHZ state is a maximally entangled state of the three-qubit system, consistent with the well-known result.

The generalized W state reads

$$|W\rangle_{\text{generalized}} = \delta_0|100\rangle + \beta_0|010\rangle + \alpha_1|001\rangle \quad\Longrightarrow\quad \begin{aligned} X_3 &= 2\,\mathrm{Re}\,\alpha_1\,\mathrm{Re}\,\delta_0 - 2\,\mathrm{Im}\,\alpha_1\,\mathrm{Im}\,\delta_0 \\ X_4 &= 2\,\mathrm{Im}\,\alpha_1\,\mathrm{Re}\,\delta_0 + 2\,\mathrm{Re}\,\alpha_1\,\mathrm{Im}\,\delta_0 \\ X_5 &= 2\,\mathrm{Re}\,\beta_0\,\mathrm{Re}\,\delta_0 - 2\,\mathrm{Im}\,\beta_0\,\mathrm{Im}\,\delta_0 \\ X_8 &= 2\,\mathrm{Im}\,\beta_0\,\mathrm{Re}\,\delta_0 + 2\,\mathrm{Re}\,\beta_0\,\mathrm{Im}\,\delta_0 \\ X_9 &= |\alpha_1|^2 + |\beta_0|^2 - |\delta_0|^2. \end{aligned} \tag{43}$$

For the W state, $X_3 = X_5 = 2X_9 = \frac{2}{3}$ and the degree of entanglement is $E = X_3^2 + X_5^2 = 8/9$, consistent with the literature [15].

4.3.5. Conjecture. A natural question is whether this construction is generalizable to systems with more than three qubits. One can imagine expanding the same formalism by always adding another square root of unity and forming the next algebra. Although this is possible via the Cayley–Dickson formalism (see the appendix), the algebras formed in this way are not alternative, and cannot be written as fibrations of spheres over sphere base spaces. The Hopf construction stops at octonions. However, the subsequent algebras, although not division, are nicely normed, which means that they have an inverse. So in principle the type of map that we give in this paper is possible. However, the map would fail in the following sense: it would be possible to map nonzero points into zeros in the base space, fact which is not possible in the maps using division algebra numbers. This is just a restatement of the fact that further algebras would have zero divisors.

However, the Cayley–Dickson construction, as well as the fact that the number of dimensions of the algebras created by this construction is identical to the number of dimensions of the qubit spaces, hints at some deeper connection between the Cayley construction and qubit states. Interestingly, this construction might be very related to the hyperdeterminant construction of Miyake and Wadati [17]. Our definition of the entanglement E in equation (39) is very similar to the hyperdeterminant construction. It would be interesting to investigate this correspondence for higher qubit states. The non-existence of Hopf maps for higher than three qubits seems to tell us that the one-qubit, two-qubit and three-qubit states are, in some sense, more special than higher qubit states. However, the richness of information that we are able to procure with the identification presented in this paper and in the paper by Mosseri and Dandoloff seems to make further investigation in this field worthy.

5. Conclusions

In this paper we analyse the three-qubit state. We give a full description of the three-qubit Hilbert space by relating it to the third and last Hopf fibration. We prove that this fibration is entanglement sensitive, that is, it can detect whether the three-qubit state is separable or entangled. Moreover, we show that one can define a quantity to describe the entanglement of the three-qubit state and the possibility of it being separable as a one-qubit $\otimes$ two-qubit state. Our results, cumulated to the results of Mosseri, show that nontrivial fibrations are a very useful tool in describing many-qubit states and their entanglement.

Acknowledgments

This paper was the result of a suggestion by S C Zhang, for which we are deeply grateful. We also acknowledge private communications with Remy Mosseri, for which we are deeply grateful. The authors would like to thank G Chapline, C H Chern, T Cuk, J Franklin, R B Laughlin, D Santiago, T-C Wei, C J Wu, J T Yard and G Zeltzer for valuable discussions. This work is supported by the NSF under grant numbers DMR-9814289 and 2FEV602, and the US Department of Energy, Office of Basic Energy Sciences under contract DE-AC03-76SF00515. The authors also acknowledge support from the Stanford Graduate Fellowship Program.

Appendix. Octonions and the last division algebra

An extensive review of octonions and division algebras is provided by Baez [16]. Real and complex numbers are used by physicists daily. Although real numbers are in a sense ‘nicer’ than complex numbers because the conjugate of a real number is itself, complex numbers bring about new and powerful properties and structure. However, they are only the first two kinds of numbers in a set of four possible structures. In a far-reaching and very deep argument, it has been proved that there are only four division algebras, in other words, there are only four vector spaces A equipped with a bilinear map $m : A \times A \rightarrow A$ called multiplication, and with a nonzero element called unit such that $m(1, a) = m(a, 1) = a$ (these properties form an algebra) and given $a, b \in A$ with $ab = 0$ then either $a = 0$ or $b = 0$ (no zero divisors—property defining the division algebra). The real and the complexes ($\mathbb{R}, \mathbb{C}$) form the first two division algebras. The third and fourth division algebras are the quaternions and the octonions ($\mathbb{Q}, \mathbb{O}$). The Cayley–Dickson construction provides a construction of the elements in $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$ which makes apparent the fact that each one fits nicely in the next. The complex numbers can be considered as a pair of real numbers (a, b); then addition can be performed componentwise whereas the multiplication rule is

$$(a, b)(c, d) = (ac - db, ad + cb) = ac - db + (ad + cb)\,\mathrm{i}. \tag{A1}$$

We can define the quaternions in a similar way: a quaternion is a pair of complex numbers (a, b), with the complex conjugation and the multiplication laws being

$$(a, b)^\star = (a^\star, -b) \qquad (a, b)(c, d) = (ac - d^\star b, bc^\star + da) = ac - d^\star b + (bc^\star + da)\,\mathrm{i}_2. \tag{A2}$$

The quaternions are non-commutative and upon expansion, can be written as $q = \mathrm{Re}\,a + \mathrm{i}_1\,\mathrm{Im}\,a + \mathrm{i}_2\,\mathrm{Re}\,b + \mathrm{i}_3\,\mathrm{Im}\,b$, $\mathrm{i}_1^2 = \mathrm{i}_2^2 = \mathrm{i}_3^2 = \mathrm{i}_1\,\mathrm{i}_2\,\mathrm{i}_3 = -1$. We can go one step further and build an octonion from a pair of quaternions (q_1, q_2), with the multiplication and conjugation laws being the same as before. The octonions are non-associative, as well as non-commutative. They are the biggest division algebra. If one continues the Cayley–Dickson construction further, by taking a pair of octonions, one discovers that the division property is lost, that is, the new numbers have zero divisors. The division algebras, including the non-associative octonions have the essential property that they are alternative, in other words

$$\forall\, a, b \in \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O} \quad \Longrightarrow \quad (aa)b = (aab) \qquad (ab)a = a(ba) \qquad (ba)a = b(aa). \tag{A3}$$

Octonions can be presented in the double quaternion format but also, equivalently, in expanded format with $\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3, \mathrm{i}_4, \mathrm{i}_6, \mathrm{i}_7$ as imaginary units (square roots of -1):

$$o = x_0 + \sum_\alpha x_m\,\mathrm{i}_m \qquad x_{0,\ldots,7} \in \mathbb{R} \qquad \mathrm{i}_1^2 = \cdots = \mathrm{i}_7^2 = -1 \tag{A4}$$

which can also be described in terms of quaternions and complex numbers as $o = \{(x_0 + x_1\,\mathrm{i}_1) + (x_2 + x_3\,\mathrm{i}_1)\,\mathrm{i}_2\} + \{(x_4 + x_7\,\mathrm{i}_1) + (x_6 - x_5\,\mathrm{i}_1)\,\mathrm{i}_2\}\,\mathrm{i}_4$. The multiplication table can be given in terms of the cycles:

$$(123) \quad (246) \quad (435) \quad (367) \quad (651) \quad (572) \quad (714) \tag{A5}$$

which read, for example $\mathrm{i}_7\,\mathrm{i}_1 = \mathrm{i}_4$, etc. The conjugate and inverse of an octonion o are

$$\bar{o} = x_0 - \sum_m x_m\,\mathrm{i}_m \qquad o^{-1} = \frac{\bar{o}}{|o|^2}. \tag{A6}$$

Another way in which an octonion o can be written is as a scalar $S(o)$ part and a vectorial $V(o)$ part:

$$S(o) = \frac{1}{2}(o + \bar{o}) = x_0 \qquad V(o) = \frac{1}{2}(o - \bar{o}) = \sum_{m=1}^{7} V_m(o)\,\mathrm{i}_m = \sum_{m=1}^{7} x_\mathrm{i}\,\mathrm{i}_m. \tag{A7}$$

An octonion o can also be written in exponential form:

$$o = |o| \exp(\theta I) \qquad \theta = \arccos\left(\frac{S(o)}{|o|}\right) \qquad I = \frac{V(o)}{|V(o)|}. \tag{A8}$$

As presented in the body of the paper, the third Hopf map is nicely presented in terms of octonions:

$$h_1 : \begin{array}{ccc} \mathbb{O} \otimes \mathbb{O} & \longrightarrow & \mathbb{O} \cup \{\infty\} \approx S^8 \\ (q_1, q_2) & \longrightarrow & h_1 = o_1 o_2^{-1} \end{array} \qquad |o_1|^2 + |o_2|^2 = 1 \tag{A9a}$$

$$h_2 : \begin{array}{ccc} \mathbb{O} \cup \{\infty\} & \longrightarrow & S^8 \\ h_1 & \longrightarrow & X_{\mathrm{i}}\,(i = 1, \ldots, 9) \end{array} \qquad \sum_{i=1}^{9} X_i^2 = 1 \tag{A9b}$$

$$h_2 \circ h_1(o_1, o_2) = X_i = \langle \sigma_{\mathrm{i}} \rangle_\Psi = (o_1^\star, o_2^\star)\sigma_{\mathrm{i}} \begin{pmatrix} o_1 \\ o_2 \end{pmatrix} \tag{A9c}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_{2,3,4,5,6,7,8} = \begin{pmatrix} 0 & \mathrm{i}_{1,2,3,4,5,6,7} \\ -\mathrm{i}_{1,2,3,4,5,6,7} & 0 \end{pmatrix} \qquad \sigma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A10}$$

are a generalization of the Pauli matrices to quaternionic space. The form of the map is identical to the form of the map presented in Mosseri *et al* [1]. However, proving that S^7 is the fibre of this map turns out to be nontrivial, since as opposed to the quaternionic case of the second Hopf map, we lose the associativity property and therefore $(o_1 o_2)o_3 \neq o_1(o_2 o_3)$ for $o_1, o_2, o_3 \in \mathbb{O}$. However, after some explicit calculations one can find out that the essential property is that the algebra be alternative. Alternativity holding, one can prove the following: $o_1^{-1}(o_1 o_2) = \left(o_1^{-1} o_1\right) o_2$ and therefore the inverse map is

$$h_1^{-1}(y) = \begin{pmatrix} \{(yd, d) \mid d \in \mathbb{O}, |yd, d| = 1\}, x \neq \infty \\ \{(c, 0) \mid c \in \mathbb{O}, |c| = 1\}, x = \infty \end{pmatrix}. \tag{A11}$$

References

[1] Mosseri R and Dandoloff R 2001 *J. Phys. A: Math. Gen.* **34** 10243
[2] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[3] Bennett C H, Brassard G, Crepeau C, Joza R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[4] Bennett C H and Wiesner S 1992 *Phys. Rev. Lett.* **69** 2881
[5] Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
[6] Ekert A K, Rarity J G, Tapster P R and Palma G M 1992 *Phys. Rev. Lett.* **69** 1293
[7] Greenberger D M, Horne M A and Zeilinger A 1989 *Bell's Theorem, Quantum Theory and Conceptions of the Universe* ed M Kafatos (Dordrecht: Kluwer) p 69
[8] Bose S, Vedral V and Knight P L 1998 *Phys. Rev.* A **57** 822
[9] Hillery M, Buzek V and Berthiaume A 1999 *Phys. Rev.* A **59** 1829
[10] Kus M and Zyczkowski K 1991 *Phys. Rev.* A **59** 032307
[11] Carteret H A and Sudbery A 2000 *J. Phys. A: Math. Gen.* **33** 4981
[12] Bennet C H *et al* 1996 *Phys. Rev.* A **53** 2046
Bennet C H *et al* 1996 *Phys. Rev.* A **54** 3824
Vedral V *et al* 1997 *Phys. Rev. Lett.* **78** 2275
Vedral V *et al* 1997 *Phys. Rev.* A **56** 4452
Vedral V *et al* 1998 *Phys. Rev.* A **57** 1619
Horodecki M *et al* 1998 *Phys. Rev. Lett.* **80** 5239
Rains E M *et al* 1999 *Phys. Rev.* A **60** 179
Hill S and Wootters W K 1997 *Phys. Rev. Lett.* **78** 5022
Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
Abouraddy A F *et al* 2001 *Phys. Rev.* A **64** 050101

[13] Lewenstein M *et al* 2001 *J. Mod. Opt.* **47** 2481

[14] Horodecki M *et al* 2000 *Preprint* quant-ph/0006071

[15] Meyer D A and Wallach N R 2002 *J. Math. Phys.* **43** 4273

[16] Baez J C 2002 *Bull. Am. Math. Soc.* **39** 145

[17] Miyake A 2003 *Phys. Rev.* A **67** 012108
Miyake A and Wadati M 2002 *Preprint* quant-ph/0212146